Received: 25 May 2018

DOI: 10.1002/mma.5334

RESEARCH ARTICLE

WILEY

Identification of an inverse source problem for time-fractional diffusion equation with random noise

Tran Ngoc Thach[1,2] | Tuan Nguyen Huy[1,2] | Pham Thi Minh Tam[1] | Mach Nguyet Minh[3] | Nguyen Huu Can[4]

[1]Environmental Sciences Lab, Institute of Computational Science and Technology, Ho Chi Minh City, Vietnam

[2]Department of Mathematics and Computer Science VNUHCM, University of Science, Ho Chi Minh City, Vietnam

[3]Department of Mathematics and Statistics, University of Helsinki, Helsinki, Finland

[4]Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

Correspondence
Nguyen Huu Can, Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, 7000 Vietnam.
Email: nguyenhuucan@tdtu.edu.vn

Communicated by: M. Kirane

MSC Classification: 35K05; 35K99; 47J06; 47H10

In this paper, we consider an inverse source problem for a time fractional diffusion equation. In general, this problem is ill posed, therefore we shall construct a regularized solution using the filter regularization method in the random noise case. We will provide appropriate conditions to guarantee the convergence of the approximate solution to the exact solution. Then, we provide examples of filters in order to obtain error estimates for their approximate solutions. Finally, we present a numerical example to show efficiency of the method.

KEYWORDS

diffusion process, fractional derivative, inverse source problem, random noise, regularization

1 | INTRODUCTION

In this paper, we consider the problem of finding a pair of functions (u, f) satisfying the following system:

$$\begin{cases} D_t^\alpha u - \triangle u = \varphi(t) f(x), & (x,t) \in (0,1)\times(0,1), \\ u_x(0,t) = u_x(1,t) = 0, & t \in (0,1), \\ u(1,t) = 0, & t\in(0,1), \\ u(x,0) = u_0(x), & x \in (0,1), \end{cases} \tag{1.1}$$

where $\varphi \in L^\infty(0,1)$ and $u_0 \in L^2(0,1)$ are given functions. Here, $D_t^\alpha u$ is the Caputo fractional derivative of order α of an absolutely continuous function u, defined as

$$D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^\alpha} ds,$$

where Γ denotes the standard Gamma function. Note that if the fractional order α tends to unity, the fractional derivative $D_t^\alpha u$ converges to the canonical first-order derivative $\frac{du}{dt}$,[1] and thus the problem 1.1 reproduces the canonical diffusion model. See, eg,[1,2] for the definition and properties of Caputo's derivative. The mathematical model 1.1 arising in dynamical systems, in control theory, electrical circuits with fractance, generalized voltage divider, viscoelasticity, fractional-order multipoles in electromagnetism, electrochemistry, and model of neurons in biology is provided in[3]; see also.[2] As we know, if $\varphi \in L^\infty(0,1), f \in L^2(\Omega)$ are given , then the direct problem of 1.1 has a unique solution in L^2 sense (see[4]). The inverse problem here is the determination of the source term $f(x)$ from the final state observation

$$u(x,1) = u_1(x). \tag{1.2}$$

where u_1 is a given function in $L^2(0,1)$. Due to the modelling as well as measurement errors, the time-dependent source term $\varphi(t)$ cannot be to obtain with infinite precision in practical applications. Instead, only observation data $\varphi_\epsilon(t)$ up to the noise level ϵ are known. Similarly, a noisy version of the final state $u_1^\epsilon \in L^2(0,1)$ is used instead of the exact one $u_1(x)$. It is known that the inverse source problem mentioned above is ill posed in general; therefore, a small perturbation in the data may cause large an error in the sought solution.

To the authors' knowledge, there are few papers for identifying an unknown source for a fractional diffusion equation by regularization method. When $\varphi = 1$, the problem with deteministic noise has been investigated using truncation method[5] and quasi-reversibilty method[6] and in some other papers.[7,8] When the time-dependent source term $\varphi(t) > 0$ is not perturbed, the fractional diffusion has been studied recently by T. Wei[9] using quasi-boundary value method. When determined measured data are replaced with random data. In random case, we have to apply our knowlegde about statictis to solve the problem. That is the reason why the random case is more difficult than the deterministic case. Until now, there are very few papers considering the random noise for inverse source problem. In,[10] the authors considered a similar problem of 1.1 with discrete random noise, ie, the input data are noised by some concerete points (called design point) . Motivated by this reason, in this paper, we consider another random data as follows:

$$\widetilde{\varphi}^\epsilon = \varphi + \epsilon\delta \qquad \widetilde{u_0}^\epsilon = u_0 + \epsilon\psi \qquad \widetilde{u_1}^\epsilon = u_1 + \epsilon\xi \tag{1.3}$$

in which ϵ corresponds to the noise level and δ, ψ, and ξ are stochatics processes, ie, bounded linear operators

$$\begin{aligned} &\xi : L^2(0,1) \to L^2(\mathcal{B},\mathcal{A},\mathcal{P}) \\ &\psi : L^2(0,1) \to L^2(\mathcal{B},\mathcal{A},\mathcal{P}) \\ &\delta : L^2(0,1) \cap L^\infty(0,1) \to L^2(\mathcal{B},\mathcal{A},\mathcal{P}) \end{aligned}$$

where $(\mathcal{B},\mathcal{A},\mathcal{P})$ is underlying probability space where $\mathcal{B}$ is sample space, $\mathcal{A}$ is σ-algebra, and $\mathcal{P}$ is probability measure. The reader is referred to[11] for more discussion on stochastic processes. The strong point of our paper is that generalized to the truncation method in.[10] To the best of our knowledge, our paper is the first investigation for the inverse source problem for fractional diffusion equation with the model 3.4.

This paper is organized as follows. In section 2, we propose a regularized scheme using a filter and prove the convergence of the approximate solution to the exact solution for a general filter. The paper ends with a couple of examples of some specific filters in section 3. Our analysis does not contain any numerical results, which we hope to obtain in a future work.

2 | MAIN RESULTS

We start by recalling the definition of the Mittag-Leffler function, a special function that plays an important role in understanding diffusion processes. For more information, see[12] and references therein.

Definition 2.1. For any $\alpha > 0$ and $\beta \in \mathbb{R}$, the two-parameter Mittag-Leffler function, a special function that plays an important role in understanding diffusion processes. For more information, see[13] and references therein

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad z \in \mathbb{C}.$$

The asymptotic growth of the Mittag-Leffler function is characterized in the following lemma.

Lemma 2.1. *There exists constants $P, B^-, B^+ > 0$ such that for all $\alpha \in (0,1)$ and $x \le 0$, it holds that*

$$\frac{B^-}{\Gamma(1-\alpha)}\frac{1}{1-x} \le E_{\alpha,1}(x) \le \frac{B^+}{\Gamma(1-\alpha)}\frac{1}{1-x}, \qquad \forall x \le 0.$$

Notice that, these estimates are uniform in α.

Proof. The proof can be found on p. 35 in Podlubny.[2] □

Lemma 2.2. *Let $u_0, u_1 \in L^2(0,1)$ and $\varphi \in L^\infty(0,1)$, and $\omega \in \mathbb{R}$. Then we have*

$$\int_0^1 f(x)\cos(\omega x)dx = \frac{\int_0^1 u_1(x)\cos(\omega x)dx - E_{\alpha,1}(-\omega^2)\int_0^1 u_0(x)\cos(\omega x)dx}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\varphi(1-s)ds}.$$

Proof. Multiplying both sides of 1.1 with $\cos(\omega x)$ and integrating from 0 to 1, we obtain

$$\frac{d^\alpha}{dt^\alpha}\int_0^1 u(x,t)\cos(\omega x)dx - \int_0^1 u_{xx}(x,t)\cos(\omega x)dx = \varphi(t)\int_0^1 f(x)\cos(\omega x)dx. \tag{2.1}$$

Integrating by parts and noting that $u(1,t) = u_x(0,t) = u_x(1,t) = 0$, we have

$$\begin{aligned}\int_0^1 u_{xx}(x,t)\cos(\omega x)dx &= u_x(x,t)\cos(\omega x)\Big|_0^1 + \omega\int_0^1 u_x(x,t)\sin(\omega x)dx \\ &= \omega\int_0^1 u_x(x,t)\sin(\omega x)dx = \omega u(x,t)\sin(\omega x)\Big|_0^1 - \omega^2\int_0^1 u(x,t)\cos(\omega x)dx \\ &= -\omega^2\int_0^1 u(x,t)\cos(\omega x)dx. \end{aligned} \tag{2.1}$$

Inserting 2.1 into 2.1, we obtain

$$\frac{d^\alpha}{dt^\alpha}\int_0^1 u(x,t)\cos(\omega x)dx = -\omega^2\int_0^1 u(x,t)\cos(\omega x)dx + \varphi(t)\int_0^1 f(x)\cos(\omega x)dx.$$

This implies that the function $h(t) = \int_0^1 u(x,t)\cos(\omega x)dx$ is a solution of the problem:

$$\begin{cases} D_t^\alpha h(t) = -\omega^2 h(t) + H(t), 0 < t < 1, \\ h(0) = h_0, \end{cases} \tag{2.3}$$

where $H(t) = \varphi(t)\int_0^1 f(x)\cos(\omega x)dx$. From[2](Theorem 3.2 on page 124), the only solution corresponding to 2.3 is given by

$$h(t) = E_{\alpha,1}(-\omega^2 t^\alpha)h_0 + \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)H(t-s)ds, \tag{2.4}$$

where $E_{\alpha,\alpha}$ is the Mittag-Leffler function of Definition 2.1. Choosing $t = 1$ and noting that $u(x,1) = u_1(x), u(x,0) = u_0(x)$, 2.4 gives

$$\int_0^1 u_1(x)\cos(\omega x)dx = E_{\alpha,1}(-\omega^2)\int_0^1 u_0(x)\cos(\omega x)dx + \int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\varphi(1-s)ds\int_0^1 f(x)\cos(\omega x)dx.$$

This completes the proof. □

In the following, we shall provide appropriate conditions so that every filter satisfying these conditions yields a meaningful numerical solution.

Assumption 2.1. Let any $\beta, \omega \in (0,\infty)$. The function $R(\beta,\omega)$ is called a filter if there exists two positive functions $K_1(\beta)$ and $K_2(\beta)$ such that

$$|R(\beta,\omega)\omega^\zeta| \le K_1(\beta) \quad \text{for all } \zeta > \frac{5}{2}, \tag{2.5}$$

$$\begin{cases} \frac{|R(\beta,\omega)-1|}{|\omega|} \le K_2(\beta), \text{ if } \omega < 1 \\ \frac{|R(\beta,\omega)-1|}{|\omega|^\mu} \le K_2(\beta), \text{ if } \omega \ge 1 \end{cases} \quad \text{for all } \mu < \frac{1}{2}. \tag{2.6}$$

Let $R_\omega(\beta) := R(\beta,\omega)$. We assume furthermore that

- $R_\omega(\beta)$ is continuous;
- $\lim_{\beta\to\infty} R_\omega(\beta) = 0$;
- $\lim_{\beta\to 0} R_\omega(\beta) = 1$;
- $R_\omega(\beta)$ is strictly decreasing function over $(0,\infty)$.

Now, we are in a position to define a regularized solution for the system 1.1-1.2.

Definition 2.2. For any given measured data $\widetilde{u_1}^\epsilon, \widetilde{u_0}^\epsilon, \widetilde{\varphi}^\epsilon$ that satisfy 3.4, we define a regularized solution for the system 1.1-1.2 as follows:

$$\widetilde{f}^\epsilon(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} R(\beta,\omega)\frac{\int_0^1 \widetilde{u_1}^\epsilon(x)\cos(\omega x)dx - E_{\alpha,1}(-\omega^2)\int_0^1 \widetilde{u_0}^\epsilon(x)\cos(\omega x)dx}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^\epsilon(1-s)ds} e^{i\omega x}d\omega, \tag{2.7}$$

where the order α belongs to $(0,1)$, and $R(\beta,\omega)$ is a filter, which defined in Assumption 2.1 .

Our main results are described in the following theorem.

Theorem 2.1. *Let $\varphi, \widetilde{\varphi}^\epsilon$ be such that $\varphi(t) \ge C_0 > 0$, $\widetilde{\varphi}^\epsilon(t) \ge \overline{C}_0 > 0$ for any $t \in [0,1]$. For any given measured data $\widetilde{u_1}^\epsilon, \widetilde{u_0}^\epsilon, \widetilde{\varphi}^\epsilon$ that satisfy 3.4, we consider the regularized solution $\widetilde{f}^\epsilon$ of the system 1.1-1.2 as in definition 2.2. We assume furthermore that*

$$\mathbb{E}\|\xi\|^2_{L^2(0,1)} = \mathbb{E}\|\psi\|^2_{L^2(0,1)} = \mathbb{E}\|\delta\|^2_{L^\infty} = 1.$$

Let K_1, K_2 be as Assumption 2.1 and assume that $\lim_{\epsilon\to 0} K_2(\beta) = \lim_{\epsilon\to 0}\epsilon K_1(\beta) = 0$. Then the function $2\widetilde{f}^\epsilon$ is an approximate solution of f and we have the following error estimate:

$$\mathbb{E}\|f - 2\widetilde{f}^\epsilon\|^2_{L^2(0,1)} \le \frac{2}{\pi}C_1^2[1 + K_1^2(\beta)]\epsilon^2 + \frac{2}{\pi}K_2^2(\beta)\left[(1+\sqrt{2})^2 + M\right]\|f\|^2_{H^1(0,1)} \tag{2.8}$$

where

$$C_1 = \frac{8\overline{C}_0 + 4\|u_0\|_{L^2(0,1)} + 4\|u_1\|_{L^2(0,1)}}{\min(C_0^2, \overline{C}_0^2)[1 - E_{\alpha,1}(-1)]}, \qquad M = \int_{\omega\ge 1}\frac{1}{\omega^2}d\omega.$$

Before prove Theorem 2.1, we state the following lemma.

Lemma 2.3. *Let δ, ξ, ψ be as Theorem 2.1. Then the following estimate holds:*

$$\mathbb{E}\left[\frac{\boldsymbol{B}(u_1, u_0, \delta, \xi, \psi)}{\min(C_0^2, \overline{C}_0^2)(1 - E_{\alpha,1}\{-1\})}\right]^2 \le C_1^2,$$

where

$$\boldsymbol{B}(u_1, u_0, \delta, \xi, \psi) = \|u_1\|_{L^2(0,1)}\|\delta\|_{L^\infty}^2 + \|u_0\|_{L^2(0,1)}\|\delta\|_{L^\infty}^2 + \overline{C}_0\|\xi\|_{L^2(0,1)} + \overline{C}_0\|\psi\|_{L^2(0,1)}. \tag{2.9}$$

Proof. Using the inequality $(a_1 + a_2 + a_3 + a_4)^2 \le 4a_1^2 + 4a_2^2 + 4a_3^2 + 4a_4^2$, we get

$$\begin{aligned}|\mathbf{B}(u_1, u_0, \delta, \xi, \psi)|^2 &= \left[\|u_1\|_{L^2(0,1)}\|\delta\|_{L^\infty}^2 + \|u_0\|_{L^2(0,1)}\|\delta\|_{L^\infty}^2 + \overline{C}_0\|\xi\|_{L^2(0,1)} + \overline{C}_0\|\psi\|_{L^2(0,1)}\right]^2 \\ &\le 4\|u_1\|_{L^2(0,1)}^2\|\delta\|_{L^\infty}^2 + 4\|u_0\|_{L^2(0,1)}^2\|\delta\|_{L^\infty}^2 + 4\overline{C}_0^2\|\xi\|_{L^2(0,1)}^2 + 4\overline{C}_0^2\|\psi\|_{L^2(0,1).}^2\end{aligned}$$

Since $\mathbb{E}\|\xi\|_{L^2(0,1)}^2 = 1$, we have by Jensen's inequality

$$(\mathbb{E}\|\xi\|)_{L^2(0,1)}^2 \le \mathbb{E}\|\xi\|_{L^2(0,1)}^2 = 1.$$

Therefore, we get $\mathbb{E}\|\xi\|_{L^2(0,1)} \le 1$. By a similar argement as above, we get $\mathbb{E}\|\psi\|_{L^2(0,1)} \le 1$ and $\mathbb{E}\|\delta\|_{L^\infty}^2 \le 1$. This implies immediately

$$\begin{aligned}\mathbb{E}(\mathbf{B}(u_1, u_0, \delta, \xi, \psi))^2 &\le 4\|u_1\|_{L^2(0,1)}^2\mathbb{E}\|\delta\|_{L^\infty}^2 + 4\|u_0\|_{L^2(0,1)}^2\mathbb{E}\|\delta\|_{L^\infty}^2 + 4\overline{C}_0^2\mathbb{E}\|\xi\|_{L^2(0,1)}^2 + 4\overline{C}_0^2\mathbb{E}\|\psi\|_{L^2(0,1)}^2 \\ &\le 8\overline{C}_0^2 + 4\|u_0\|_{L^2(0,1)}^2 + 4\|u_1\|_{L^2(0,1)}^2.\end{aligned} \tag{2.10}$$

□

Proof. Let the function $\widetilde{f}$ be defined as

$$\widetilde{f} = \begin{cases} \frac{1}{2}f(x) & x \in (0,1), \\ \frac{1}{2}f(-x) & x \in (-1,0), \\ 0 & \text{otherwise}. \end{cases} \tag{2.11}$$

Since $\widetilde{f}$ is odd function, we know that

$$\int_{-\infty}^{\infty} \widetilde{f}(x)\sin(\omega x)\Big) dx = 0.$$

The Fourier transform of $\widetilde{f}$ is given by

$$\begin{aligned}\mathcal{F}(\widetilde{f})(\omega) &= \int_{-\infty}^{\infty} \widetilde{f}(x)e^{-i\omega x}dx \\ &= \int_{-\infty}^{\infty} \left[\widetilde{f}(x)\cos(\omega x) - i\widetilde{f}(x)sin(\omega x)\right] dx \\ &= \int_0^1 2\widetilde{f}(x)\cos(\omega x)dx = \int_0^1 f(x)\cos(\omega x)dx \\ &= \frac{\int_0^1 u_1\cos(\omega x)dx - E_{\alpha,1}(-\omega^2)\int_0^1 u_0\cos(\omega x)dx}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\varphi(1-s)ds},\end{aligned} \tag{2.12}$$

which we have used Lemma 2.2 in the last equality. Since 3.8, we obtain the Fourier transform of $\widetilde{f}^\epsilon$ as follows:

$$\begin{aligned}\mathcal{F}(\widetilde{f}^\epsilon) &= \mathcal{F}\left\{\mathcal{F}^{-1}\left[R(\beta,\omega)\frac{\int_0^1 \widetilde{u_1}^\epsilon(x)\cos(\omega x)dx - E_{\alpha,1}(-\omega^2)\int_0^1 \widetilde{u_0}^\epsilon(x)\cos(\omega x)dx}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^\epsilon(1-s)ds}\right]\right\} \\ &= R(\beta,\omega)\frac{\int_0^1 \widetilde{u_1}^\epsilon(x)\cos(\omega x)dx - E_{\alpha,1}(-\omega^2)\int_0^1 \widetilde{u_0}^\epsilon(x)\cos(\omega x)dx}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^\epsilon(1-s)ds}.\end{aligned} \tag{2.13}$$

Let us define the following function

$$v_\beta(\omega) = R(\beta,\omega)\frac{\int_0^1 u_1\cos(\omega x)dx - E_{\alpha,1}(-\omega^2)\int_0^1 u_0\cos(\omega x)dx}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\varphi(1-s)ds}. \tag{2.14}$$

For any $\omega\in\mathbb{R}$, using two above equalities, we obtain

$$\begin{aligned} v_\beta(\omega) - \mathcal{F}(\widetilde{f}^\epsilon)(\omega) &= R(\beta,\omega)\left[\frac{\int_0^1 u_1\cos(\omega x)dx - E_{\alpha,1}(-\omega^2)\int_0^1 u_0\cos(\omega x)dx}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\varphi(1-s)ds}\right.\\ &\qquad\left.-\frac{\int_0^1 \widetilde{u_1}^\epsilon(x)\cos(\omega x)dx - E_{\alpha,1}(-\omega^2)\int_0^1 \widetilde{u_0}^\epsilon(x)\cos(\omega x)dx}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^\epsilon(1-s)ds}\right]\\ &= R(\beta,\omega)\left[\frac{\int_0^1 u_1(x)\cos(\omega x) - E_{\alpha,1}(-\omega^2)\int_0^1 u_0\cos(\omega x)d\omega}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\varphi(1-s)ds}\right.\\ &\qquad-\frac{\int_0^1 u_1(x)\cos(\omega x) - E_{\alpha,1}(-\omega^2)\int_0^1 u_0\cos(\omega x)d\omega}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^\epsilon(1-s)ds}\\ &\qquad+\frac{\int_0^1 u_1(x)\cos(\omega x) - E_{\alpha,1}(-\omega^2)\int_0^1 u_0\cos(\omega x)d\omega}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^\epsilon(1-s)ds}\\ &\qquad\left.-\frac{\int_0^1 \widetilde{u_1}^\epsilon(x)\cos(\omega x) - E_{\alpha,1}(-\omega^2)\int_0^1 \widetilde{u_0}^\epsilon\cos(\omega x)d\omega}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^\epsilon(1-s)ds}\right]. \end{aligned} \tag{2.15}$$

Using 3.4, we obtain

$$\begin{aligned} &v_\beta(\omega) - \mathcal{F}(\widetilde{f}^\epsilon)(\omega)\\ &= R(\beta,\omega)\left\{\frac{\left(\int_0^1 u_1(x)\cos(\omega x) - E_{\alpha,1}(-\omega^2)\int_0^1 u_0\cos(\omega x)d\omega\right)\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)(\widetilde{\varphi}^\epsilon(1-s)-\varphi(1-s))ds}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^\epsilon(1-s)ds\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\varphi(1-s)ds}\right.\\ &\quad\left.+\frac{\int_0^1(u_1(x)-\widetilde{u_1}^\epsilon(x))\cos(\omega x)ds}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^\epsilon(1-s)ds} - E_{\alpha,1}(-\omega^2)\frac{\int_0^1(u_0(x)-\widetilde{u_0}^\epsilon(x))\cos(\omega x)ds}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^\epsilon(1-s)ds}\right\}\\ &= R(\beta,\omega)\left\{-\underbrace{\frac{\epsilon\left(\int_0^1 u_1(x)\cos(\omega x) - E_{\alpha,1}(-\omega^2)\int_0^1 u_0\cos(\omega x)d\omega\right)\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\delta ds}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^\epsilon(1-s)ds\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\varphi(1-s)ds}}_{I_1(\omega)}\right.\\ &\quad\left.+\underbrace{\frac{\epsilon\int_0^1\xi\cos(\omega x)ds}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^\epsilon(1-s)ds}}_{I_2(\omega)} - \underbrace{E_{\alpha,1}(-\omega^2)\frac{\epsilon\int_0^1\psi\cos(\omega x)ds}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^\epsilon(1-s)ds}}_{I_3(\omega)}\right\}. \end{aligned} \tag{2.16}$$

This implies immediately

$$|v_\beta(\omega) - \mathcal{F}(\widetilde{f}^\epsilon)(\omega)| \le R(\beta,\omega)\left(|I_1(\omega)| + |I_2(\omega)| + I_3(\omega)|\right).$$

Step 1. Estimate $I_1(\omega)$. Using the fact that $\varphi(t) \geq C_0 > 0,\ \widetilde{\varphi}^{\epsilon}(t) \geq \overline{C}_0 > 0$ for any $t \in [0,1]$, we have

$$|I_1(\omega)| = \frac{\epsilon \left| \int_0^1 u_1(x)\cos(\omega x) - E_{\alpha,1}(-\omega^2)\int_0^1 u_0 \cos(\omega x) d\omega \right| \int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)|\delta| ds}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^{\epsilon}(1-s)ds \int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\varphi(1-s)ds}$$
$$\leq \frac{\epsilon \left(\int_0^1 |u_1(x)||\cos(\omega x)| + E_{\alpha,1}(-\omega^2)\int_0^1 |u_0||\cos(\omega x)| d\omega \right) \int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)|\delta| ds}{C_0\overline{C}_0 \left(\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)ds \right)^2}. \tag{2.17}$$

Due to Lemma 2.4[14] and applying the Hölder inequality, we deduce that

$$|I_1(\omega)| \leq \frac{\epsilon \left(\|u_1\|_{L^2(0,1)} + \|u_0\|_{L^2(0,1)} \right) \|\delta\|_{L^\infty} \int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)ds}{C_0\overline{C}_0 \left(\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)ds \right)^2}$$
$$= \frac{\epsilon\omega^2 \left(\|u_1\|_{L^2(0,1)} + \|u_0\|_{L^2(0,1)} \right) \|\delta\|^2_{L^\infty}}{C_0\overline{C}_0(1-E_{\alpha,1}(-1))}. \tag{2.18}$$

Step 2. Estimate $I_2(\omega)$. We have

$$|I_2(\omega)| = \frac{\epsilon \left| \int_0^1 \xi \cos(\omega x) ds \right|}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^{\epsilon}(1-s)ds} \leq \frac{\epsilon\omega^2\|\xi\|_{L^2(0,1)}}{\overline{C}_0(1-E_{\alpha,1}(-1))}$$
$$\leq \begin{cases} \frac{\epsilon\omega^2\|\xi\|_{L^2(0,1)}}{\overline{C}_0(1-E_{\alpha,1}(-1))}, & \text{if} \quad |\omega| \geq 1, \\ \frac{\epsilon\|\xi\|_{L^2(0,1)}}{\overline{C}_0(1-E_{\alpha,1}(-1))}, & \text{if} \quad |\omega| < 1. \end{cases} \tag{2.19}$$

Step 3. Estimate $I_3(\omega)$. We bound $|I_3(\omega)|$ as follows:

$$|I_3(\omega)| = E_{\alpha,1}(-\omega^2)\frac{\epsilon \left| \int_0^1 \psi \cos(\omega x) ds \right|}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\widetilde{\varphi}^{\epsilon}(1-s)ds} \leq \frac{E_{\alpha,1}(-1)\omega^2\epsilon\|\psi\|_{L^2(0,1)}ds}{\overline{C}_0(1-E_{\alpha,1}(-1))}$$
$$\leq \begin{cases} \frac{\epsilon\omega^2\|\psi\|_{L^2(0,1)}}{\overline{C}_0(1-E_{\alpha,1}(-1))}, & \text{if} \quad |\omega| \geq 1, \\ \frac{\epsilon\|\psi\|_{L^2(0,1)}}{\overline{C}_0(1-E_{\alpha,1}(-1))}, & \text{if} \quad |\omega| < 1, \end{cases}. \tag{2.20}$$

From the above observations, we can conclude that

$$\left| v_\beta(\omega) - \mathcal{F}(\widetilde{f}^{\epsilon})(\omega) \right| \leq \begin{cases} \frac{\epsilon\omega^2 R(\beta,\omega)\left(\mathbf{B}(u_1,u_0,\delta,\xi,\psi)\right)}{\min(C_0^2,\overline{C}_0^2)(1-E_{\alpha,1}(-1))}, & \text{if} \quad |\omega| \geq 1, \\ \frac{\epsilon R(\beta,\omega)\left(\mathbf{B}(u_1,u_0,\delta,\xi,\psi)\right)}{\min(C_0^2,\overline{C}_0^2)(1-E_{\alpha,1}(-1))}, & \text{if} \quad |\omega| < 1, \end{cases} \tag{2.21}$$

where $\mathbf{B}$ is given by 2.9. Using Lemma 2.3, we have

- If $|\omega| \geq 1$ then

$$\left| v_\beta(\omega) - \mathcal{F}(\widetilde{f}^{\epsilon})(\omega) \right|^2 \leq \frac{R(\beta,\omega)^2\epsilon^2\omega^4(A_1(u_1,u_0,\delta,\xi,\psi))^2}{\min(C_0^4,\overline{C}_0^4)(1-E_{\alpha,1}(-1))^2}. \tag{2.22}$$

- If $|\omega| < 1$ then

$$\left| v_\beta(\omega) - \mathcal{F}(\widetilde{f}^{\epsilon})(\omega) \right|^2 \leq \frac{R(\beta,\omega)^2\epsilon^2(A_1(u_1,u_0,\delta,\xi,\psi))^2}{\min(C_0^4,\overline{C}_0^4)(1-E_{\alpha,1}(-1))^2}. \tag{2.23}$$

Since $\int_{|\omega|>1}\frac{1}{|\omega|^{2\zeta-4}}=M<\infty$, we have the following estimation:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}|v_\beta(\omega)-\mathcal{F}(\widetilde{f}^\epsilon)(\omega)|^2d\omega\\
&=\frac{\epsilon^2\,|\mathbf{B}(u_1,u_0,\delta,\xi,\psi)|^2}{\min(C_0^4,\overline{C}_0^{-4})(1-E_{\alpha,1}(-1))^2}\int_{|\omega|>1}R(\beta,\omega)^2\omega^4d\omega+\frac{\epsilon^2\,|\mathbf{B}(u_1,u_0,\delta,\xi,\psi)|^2}{\min(C_0^4,\overline{C}_0^{-4})(1-E_{\alpha,1}(-1))^2}\int_{|\omega|\le1}R(\beta,\omega)^2d\omega\\
&=\frac{\epsilon^2\,|\mathbf{B}(u_1,u_0,\delta,\xi,\psi)|^2}{\min(C_0^4,\overline{C}_0^{-4})(1-E_{\alpha,1}(-1))^2}\int_{|\omega|>1}\frac{R(\beta,\omega)^2|\omega|^{2\zeta}}{|\omega|^{2\zeta-4}}d\omega+K_1^2(\beta)\frac{\epsilon^2\,|\mathbf{B}(u_1,u_0,\delta,\xi,\psi)|^2}{\min(C_0^4,\overline{C}_0^{-4})(1-E_{\alpha,1}(-1))^2}\int_{|\omega|\le1}d\omega\\
&=\frac{K_1^2(\beta)\epsilon^2\,|\mathbf{B}(u_1,u_0,\delta,\xi,\psi)|^2}{\min(C_0^4,\overline{C}_0^{-4})(1-E_{\alpha,1}(-1))^2}\int_{|\omega|>1}\frac{1}{|\omega|^{2\zeta-4}}d\omega+\frac{\epsilon^2\,|\mathbf{B}(u_1,u_0,\delta,\xi,\psi)|^2}{\min(C_0^4,\overline{C}_0^{-4})(1-E_{\alpha,1}(-1))^2}\\
&=\frac{MK_1^2(\beta)\epsilon^2\,|\mathbf{B}(u_1,u_0,\delta,\xi,\psi)|^2}{\min(C_0^4,\overline{C}_0^{-4})(1-E_{\alpha,1}(-1))^2}+\frac{\epsilon^2\,|\mathbf{B}(u_1,u_0,\delta,\xi,\psi)|^2}{\min(C_0^4,\overline{C}_0^{-4})(1-E_{\alpha,1}(-1))^2}\\
&=\frac{M(K_1^2(\beta)+1)\epsilon^2\,|\mathbf{B}(u_1,u_0,\delta,\xi,\psi)|^2}{\min(C_0^4,\overline{C}_0^{-4})(1-E_{\alpha,1}(-1))^2}.
\end{aligned}
\tag{2.24}
$$

This implies that

$$
\mathbb{E}\left(\int_{-\infty}^{\infty}|v_\beta(\omega)-\mathcal{F}(\widetilde{f}^\epsilon)(\omega)|^2\right)=\frac{(1+K_1^2(\beta))\epsilon^2\mathbb{E}\,|\mathbf{B}(u_1,u_0,\delta,\xi,\psi)|^2}{\min(C_0^4,\overline{C}_0^{-4})(1-E_{\alpha,1}(-1))^2}=C_1^2(1+K_1^2(\beta))\epsilon^2. \tag{2.25}
$$

Then, we estimate $\mathbb{E}\|v_\beta-\mathcal{F}(\widetilde{f})\|^2$. Subtracting 2.12 from 2.14, and by taking the square of the obtained result, we have

$$
|v_\beta-\mathcal{F}(\widetilde{f})|^2=(R(\beta,\omega)-1)^2\left|\frac{\int_0^1u_1\cos(\omega x)dx-E_{\alpha,1}(-\omega^2)\int_0^1u_0\cos(\omega x)dx}{\int_0^1s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2s^\alpha)\varphi(1-s)ds}\right|^2. \tag{2.26}
$$

This leads to

$$
\begin{aligned}
\int_{\mathbb{R}}\left|v_\beta(\omega)-\mathcal{F}(\widetilde{f})(\omega)\right|^2d\omega&\le\int_{\omega<1}(R(\beta,\omega)-1)^2\left|\frac{\int_0^1u_1\cos(\omega x)dx-E_{\alpha,1}(-\omega^2)\int_0^1u_0\cos(\omega x)dx}{\int_0^1s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2s^\alpha)\varphi(1-s)ds}\right|^2d\omega\\
&\quad+\int_{\omega\ge1}(R(\beta,\omega)-1)^2\left|\frac{\int_0^1u_1\cos(\omega x)dx-E_{\alpha,1}(-\omega^2)\int_0^1u_0\cos(\omega x)dx}{\int_0^1s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2s^\alpha)\varphi(1-s)ds}\right|^2d\omega\\
&\le K_2^2(\beta)\int_{\omega<1}\omega^2\left|\frac{\int_0^1u_1\cos(\omega x)dx-E_{\alpha,1}(-\omega^2)\int_0^1u_0\cos(\omega x)dx}{\int_0^1s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2s^\alpha)\varphi(1-s)ds}\right|^2d\omega\\
&\quad+K_2^2(\beta)\int_{\omega\ge1}\omega^{2\mu}\left|\frac{\int_0^1u_1\cos(\omega x)dx-E_{\alpha,1}(-\omega^2)\int_0^1u_0\cos(\omega x)dx}{\int_0^1s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2s^\alpha)\varphi(1-s)ds}\right|^2d\omega.
\end{aligned}
\tag{2.27}
$$

Since $f\in H^1(0,1)$, we have

$$
\mathcal{F}(\widetilde{f})(\omega)=\int_0^1f(x)\cos(\omega x)dx=f(1)\frac{\sin\omega}{\omega}-\frac{1}{\omega}\int_0^1f'(x)\sin(\omega x)dx,\qquad\forall\omega\neq0. \tag{2.28}
$$

It implies that

$$
|\mathcal{F}(\widetilde{f})(\omega)|\le\frac{|f(1)|}{|\omega|}+\frac{\|f'\|}{|\omega|}.
$$

Applying Lagrange mean value theorem, we deduce that there exits $x_0 \in [0,1]$ such that $\int_0^1 f(x)dx = (1-0)f(x_0) = f(x_0)$. Furthermore, we get

$$f(1) = f(x_0) + \int_{x_0}^1 f'(x)dx = \int_0^1 f(x)dx + \int_{x_0}^1 f'(x)dx. \tag{2.29}$$

Recall that $H^1(0,1)$ is the space of the function f such that f and f' belong to $L^2(0,1)$ with the norm defined by

$$\|f\|^2_{H^1(0,1)} = \|f\|^2_{L^2(0,1)} + \|f'\|^2_{L^2(0,1)}.$$

It follows from 2.29 that

$$|f(1)| \le \int_0^1 (|f(x)| + |f'(x)|)dx \le \sqrt{2\int_0^1 \left(|f(x)|^2 + |f'(x)|^2\right)dx} \le \sqrt{2}\|f\|_{H^1(0,1)}. \tag{2.30}$$

Hence,

$$|\mathcal{F}(\widetilde{f})(\omega)| = \left|\frac{\int_0^1 u_1\cos(\omega x)dx - E_{\alpha,1}(-\omega^2)\int_0^1 u_0\cos(\omega x)dx}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\varphi(1-s)ds}\right| \le \frac{1+\sqrt{2}}{|\omega|}\|f\|_{H^1(0,1)}, \qquad \forall \omega \ne 0. \tag{2.31}$$

Since $\mu < \frac{1}{2}$, we know that $\int_{\omega\ge 1}\frac{\omega^{2\mu}}{\omega^2}d\omega = M < \infty$. This leads to

$$\begin{aligned}\int_{\mathbb{R}} \left|v_\beta(\omega) - \mathcal{F}(\widetilde{f})(\omega)\right|^2 d\omega &\le K_2^2(\beta)(1+\sqrt{2})^2\|f\|^2_{H^1(0,1)}\int_{\omega<1}\frac{\omega^2}{\omega^2}d\omega \\ &\quad + K_2^2(\beta)(1+\sqrt{2})^2\|f\|^2_{H^1(0,1)}\int_{\omega\ge 1}\frac{\omega^{2\mu}}{\omega^2}d\omega \\ &\le K_2^2(\beta)(1+\sqrt{2})^2(1+M)\|f\|^2_{H^1(0,1)}.\end{aligned} \tag{2.32}$$

Using Plancherel theorem, we obtain

$$\begin{aligned}\mathbb{E}\|f - 2\widetilde{f}^\epsilon\|^2_{L^2(0,1)} &\le \frac{2}{\pi}\left(\|\mathcal{F}(\widetilde{f}) - v_\beta\|^2 + \mathbb{E}\|v_\beta - \mathcal{F}(\widetilde{f}^\epsilon)\|^2\right) \\ &\le \frac{2}{\pi}C_1^2(1+K_1^2(\beta))\epsilon^2 + \frac{2}{\pi}K_2^2(\beta)\left[(1+\sqrt{2})^2 + M\right]\|f\|^2_{H^1(0,1)}.\end{aligned} \tag{2.33}$$

□

Remark 2.1. Our method in this paper can be applied for solving the time fractional diffusion for any high dimensions (in 2D and 3D).

Remark 2.2. In our theorem, we assume that f belongs to $H^1(0,1)$. This assumption is delicate since f is unknown data. Assumme that some other conditions on given data of the problem. For $\eta > 0$, we set the following space:

$$\mathbf{H}^\eta := \left\{ v \in L^2(0,1) : \int_{-\infty}^{+\infty} \omega^{2\eta}\left(\int_0^1 v(x)\cos(\omega x)dx\right)^2 d\omega < \infty \right\}. \tag{2.34}$$

If $u_0 \in \mathbf{H}^\gamma,\ u_1 \in \mathbf{H}^{\gamma+1}$ with $\gamma > \frac{1+2\mu}{4}$, we will obtain the convergence rate. Indeed, since 2.31, we get

$$\begin{aligned}\left|\mathcal{F}(\widetilde{f})(\omega)\right| &= \left|\frac{\int_0^1 u_1\cos(\omega x)dx - E_{\alpha,1}(-\omega^2)\int_0^1 u_0\cos(\omega x)dx}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\varphi(1-s)ds}\right| \\ &\le \frac{\omega^2}{C_0(1-E_{\alpha,1}(-1))}\left[\int_0^1 u_1\cos(\omega x)dx - E_{\alpha,1}(-\omega^2)\int_0^1 u_0\cos(\omega x)dx\right].\end{aligned} \tag{2.35}$$

Hence,

$$\left|\mathcal{F}(\widetilde{f})(\omega)\right| \le Cw^{-2\gamma}\int_{-\infty}^{+\infty}\omega^{2\gamma+2}\left(\int_0^1 u_1(x)\cos(\omega x)dx\right)^2 d\omega + Cw^{-2\gamma}\int_{-\infty}^{+\infty}\omega^{2\gamma}\left(\int_0^1 u_0(x)\cos(\omega x)dx\right)^2, \quad (2.36)$$

where C indicates a constant, which only depends on C_0, α. This implies that

$$\left|\mathcal{F}(\widetilde{f})(\omega)\right|^2 \le Cw^{-4\gamma}\left(\|u_1\|^2_{\mathbf{H}^{\gamma+1}} + \|u_0\|^2_{\mathbf{H}^{\gamma}}\right). \quad (2.37)$$

The right hand side of 2.27 is bounded by

$$\text{RHS of}(2.28) \le C\left[K_2^2(\beta)\int_{\omega<1} w^{2-4\gamma}d\omega + K_2^2(\beta)\int_{\omega\ge1} w^{2\mu-4\gamma}d\omega\right]\left(\|u_1\|^2_{\mathbf{H}^{\gamma+1}} + \|u_0\|^2_{\mathbf{H}^{\gamma}}\right). \quad (2.38)$$

The right hand side of the latter inequality is convergent if and only if the integral $\int_{\omega\ge1} w^{2\mu-4\gamma}d\omega$ is convergent. Since $4\gamma - 2\mu > 1$, we obtain the convergence rate, which is similar to 2.32. In the future work, we try to consider the regularized solution of possible f in the class H^1.

Corollary 2.1. *Let*

$$R(\beta,\omega) = \begin{cases} 1 & |\omega| \le \beta^{-\frac{1}{\zeta}}, \\ 0 & |\omega| > \beta^{-\frac{1}{\zeta}}, \end{cases} \quad (2.39)$$

where $\zeta > \frac{5}{2}$. Then, $R(\beta,\omega)$ satisfies 2.5 and 2.6. Here, $K_1(\beta) = \frac{1}{\beta}$,

$$K_2(\beta) = \max\left\{\beta^{\frac{1}{\zeta}}, \beta^{\frac{\mu}{\zeta}}\right\}, \quad (2.40)$$

where $\mu < \frac{1}{2}$ and $\beta = \epsilon^{\lambda}$ with $\lambda \in (0,1)$.

Proof. The constraint 2.5 is obtained as

$$R(\beta,\omega)|\omega|^{\zeta} = R(\beta,\omega) = \begin{cases} |\omega|^{\zeta} & |\omega| \le \beta^{-\frac{1}{\zeta}} \\ 0 & |\omega| > \beta^{-\frac{1}{\zeta}} \end{cases} < \begin{cases} \beta^{-1} & |\omega| \le \beta^{-\frac{1}{\zeta}} \\ 0 & |\omega| > \beta^{-\frac{1}{\zeta}} \end{cases} < \beta^{-1} = K_1(\beta). \quad (2.41)$$

The constraint 2.6 is obtained as, we consider $|\omega| < 1$

$$\frac{|R(\beta,\omega)-1|}{|\omega|} = \begin{cases} 0 & |\omega| \le \beta^{-\frac{1}{\zeta}} \\ |\omega|^{-1} & |\omega| > \beta^{-\frac{1}{\zeta}}. \end{cases} \quad (2.42)$$

Since $|\omega| > \beta^{-\frac{1}{\zeta}}$, we have $|\omega|^{-1} < \beta^{\frac{1}{\zeta}}$, so

$$\frac{|R(\beta,\omega)-1|}{|\omega|} \le \begin{cases} 0 & |\omega| \le \beta^{-\frac{1}{\zeta}} \\ \beta^{\frac{1}{\zeta}} & |\omega| > \beta^{-\frac{1}{\zeta}} \end{cases} \le \beta^{\frac{1}{\zeta}}. \quad (2.43)$$

If $|\omega| \ge 1$, we have

$$\frac{|R(\beta,\omega)-1|}{|\omega|^{\mu}} = \begin{cases} 0 & |\omega| \le \beta^{-\frac{1}{\zeta}} \\ |\omega|^{-\mu} & |\omega| > \beta^{-\frac{1}{\zeta}}. \end{cases} \quad (2.44)$$

Since $|\omega| > \beta^{-\frac{1}{\zeta}}$, we have $|\omega|^{-\mu} < \beta^{\frac{\mu}{\zeta}}$, so

$$\frac{|R(\beta,\omega)-1|}{|\omega|^{\mu}} \le \begin{cases} 0 & |\omega| \le \beta^{-\frac{1}{\zeta}} \\ \beta^{\frac{\mu}{\zeta}} & |\omega| > \beta^{-\frac{1}{\zeta}} \end{cases} \le \beta^{\frac{\mu}{\zeta}}. \quad (2.45)$$

Let us choose $K_2(\beta) = \max\{\beta^{\frac{\mu}{\zeta}}, \beta^{\frac{1}{\zeta}}\}$ then the proof of this corollary is completed. □

Lemma 2.1. *Let μ, β be positive constants, and $\zeta > \mu$. Let $G(z)$ be a function defined by*

$$G(z) = \frac{\beta z^{\zeta-\mu}}{1+\beta z^{\zeta}} \qquad z > 0. \tag{2.46}$$

Then, $G(z) \leq \frac{\mu}{\zeta}\left[\frac{\zeta-\mu}{\mu}\right]^{1-\frac{\mu}{\zeta}} \beta^{\frac{\mu}{\zeta}}$.

Proof. It is easy to see that

$$\lim_{z\to 0} G(z) = \lim_{z\to 0} \frac{\beta z^{\zeta-\mu}}{1+\beta z^{\zeta}} = \lim_{z\to 0} \frac{\beta}{z^{\mu-\zeta}+\beta z^{\mu}} \leq \lim_{z\to 0} \frac{\beta}{z^{\mu-\zeta}} = 0. \tag{2.47}$$

$$\lim_{z\to \infty} G(z) = \lim_{z\to \infty} \frac{\beta z^{\zeta-\mu}}{1+\beta z^{\zeta}} = \lim_{z\to \infty} \frac{\beta}{z^{\mu-\zeta}+\beta z^{\mu}} \leq \lim_{z\to \infty} \frac{1}{z^{\mu}} = 0. \tag{2.48}$$

By taking the derivative of G with respect to z, we have

$$G'(z) = \frac{\left[(\zeta-\mu) - \mu\beta z^{\zeta}\right] \beta z^{\zeta-\mu-1}}{(1+\beta z^{\zeta})^2}. \tag{2.49}$$

The function $G(z)$ attains maximun value at $z = z_0$, which satisfies $G^{'}(z) = 0$. Solving $G^{'}(z_0)$, we get $z_0 = \sqrt[\zeta]{\frac{\zeta-\mu}{\mu}}\beta^{\frac{-1}{\zeta}}$. Hence,

$$G(z) \leq G(z_0) = \frac{\mu}{\zeta}\left(\frac{\zeta-\mu}{\mu}\right)^{1-\frac{\mu}{\zeta}} \beta^{\frac{\mu}{\zeta}}. \tag{2.50}$$

Then, we derive constraint 2.6 as follow. With $|\omega| \leq 1$, we have

$$\frac{|R(\beta,\omega)-1|}{|\omega|} = \frac{\beta|\omega|^{\zeta-1}}{1+\beta|\omega|^{\zeta}}. \tag{2.51}$$

□

Remark 2.3. With $R(\beta,\omega)$ and β as in Corollary 3.1, applying Theorem 2.1, we conclude that the expectation of the error between the regularized solution and the exact solution $\mathbb{E}\|f - 2\widetilde{f}^{\epsilon}\|^2_{L^2(0,1)}$ is of order

$$\max\left\{\epsilon^{\frac{2\lambda}{\zeta}}, \epsilon^{\frac{2\lambda\mu}{\zeta}}, \epsilon^{2-2\lambda}\right\}$$

for any $0 < \lambda < 1$.

Corollary 2.2. *Let ζ, μ be as Corollary 2.1. Let*

$$R(\beta,\omega) = \frac{1}{1+\beta|\omega|^{\zeta}}. \tag{2.52}$$

Then, $R(\beta,\omega)$ satisfies 2.5 and 2.6 with $\beta(\epsilon) = \epsilon^{\lambda}$ for $\lambda \in (0,1)$. Here, $K_1(\beta) = \beta^{-1}$,

$$K_2(\beta) = \max\left\{\frac{\mu}{\zeta}\left(\frac{\zeta-\mu}{\mu}\right)^{1-\frac{\mu}{\zeta}}\beta^{\frac{\mu}{\zeta}}; \quad \frac{(\zeta-1)^{1-\frac{1}{\zeta}}}{\zeta}\beta^{\frac{1}{\zeta}}\right\}. \tag{2.53}$$

Proof. The constraint 2.5 is obtained as

$$|R(\beta,\omega)||\omega|^{\zeta} = \frac{|\omega|^{\zeta}}{1+\beta|\omega|^{\zeta}} \leq \frac{1}{\frac{1}{|\omega|^{\zeta}}+\beta} \leq \frac{1}{\beta}. \tag{2.54}$$

We derive constraint 2.6 as follow. With $|\omega| > 1$, we have

$$\frac{|R(\beta,\omega)-1|}{|\omega|^{\mu}} = \frac{\beta|\omega|^{\zeta-\mu}}{1+\beta|\omega|^{\zeta}}. \tag{2.55}$$

Let $H(z) = \frac{\beta z^{\zeta-1}}{1+\beta z^{\zeta}}$ where $z > 0$. It is easy to see that

$$\lim_{z\to\infty} H(z) = \lim_{z\to\infty} \frac{\beta z^{\zeta-1}}{1+\beta z^{\zeta}} = \lim_{z\to\infty} \frac{\beta}{z^{-\zeta+1}+\beta z} \leq \lim_{z\to\infty} \frac{1}{z} = 0. \tag{2.56}$$

$$\lim_{z\to 0} H(z) = \lim_{z\to 0} \frac{\beta z^{\zeta-1}}{1+\beta z^{\zeta}} = \lim_{z\to 0} \frac{\beta}{z^{-\zeta+1}+\beta z} \leq \lim_{z\to 0} \frac{\beta}{z^{-\zeta+1}} = 0. \tag{2.57}$$

By taking the derivative of H with respect to z, we have

$$H'(z) = \frac{((\zeta-1)-\beta z^{\zeta})\beta z^{\zeta-2}}{(1+\beta z^{\zeta})^2}. \tag{2.58}$$

The function $H(z)$ attains maximum value at $z = z_0$, which satisfies $H'(z) = 0$. Solving $H'(z_0)$, we get $z_0 = \sqrt[\zeta]{\frac{\zeta-1}{\beta}}$. Hence,

$$H(z) \leq H(z_0) = \frac{(\zeta-1)^{1-\frac{1}{\zeta}}}{\zeta}\beta^{\frac{1}{\zeta}}. \tag{2.59}$$

□

Corollary 2.3. *Let ζ, μ be as Corollary 2.1. Let*

$$R(\beta,\omega) = \frac{1}{1+\beta\omega^{2\zeta}}. \tag{2.60}$$

Then, $R(\beta,\omega)$ satisfies 2.5 and 2.6. Here, $K_1(\beta) = \frac{1}{\sqrt{2\beta}}$,

$$K_2(\beta) = \max\left\{ \frac{\mu}{2\zeta}\left(\frac{2\zeta-\mu}{\mu}\right)^{1-\frac{\mu}{2\zeta}}\beta^{\frac{\mu}{2\zeta}}; \qquad \frac{(2\zeta-1)^{1-\frac{1}{2\zeta}}}{2\zeta}\beta^{\frac{1}{2\zeta}} \right\}, \tag{2.61}$$

and $\beta = \epsilon^{2\lambda}$ with $\lambda \in (0,1)$.

Proof. The constraint 2.5 is obtained as

$$|R(\beta,\omega)||\omega|^{\zeta} = \frac{|\omega|^{\zeta}}{1+\beta|\omega|^{2\zeta}} = \frac{1}{\frac{1}{|\omega|^{\zeta}}+\beta|\omega|^{\zeta}} \leq \frac{1}{\sqrt{2\beta}} = K_1(\beta). \tag{2.62}$$

We derive constraint 2.6 as follow. With $|\omega| > 1$, we have

$$\frac{|R(\beta,\omega)-1|}{|\omega|^{\mu}} = \frac{\beta|\omega|^{2\zeta-\mu}}{1+\beta|\omega|^{2\zeta}}. \tag{2.63}$$

Let $G(z)$ be a function defined by

$$G(z) = \frac{\beta z^{2\zeta-\mu}}{1+\beta z^{2\zeta}} \qquad z > 0. \tag{2.64}$$

It is easy to see that

$$\lim_{z\to 0} G(z) = \lim_{z\to 0} \frac{\beta z^{2\zeta-\mu}}{1+\beta z^{2\zeta}} = \lim_{z\to 0} \frac{\beta}{z^{\mu-2\zeta}+\beta z^{\mu}} \leq \lim_{z\to 0} \frac{\beta}{z^{\mu-2\zeta}} = 0. \tag{2.65}$$

$$\lim_{z\to\infty} G(z) = \lim_{z\to\infty} \frac{\beta z^{2\zeta-\mu}}{1+\beta z^{2\zeta}} = \lim_{z\to\infty} \frac{\beta}{z^{\mu-2\zeta}+\beta z^{\mu}} \le \lim_{z\to\infty} \frac{1}{z^{\mu}} = 0. \tag{2.66}$$

By taking the derivative of G with respect to z, we have

$$G'(z) = \frac{\left[(2\zeta-\mu) - \mu\beta z^{2\zeta}\right] \beta z^{2\zeta-\mu-1}}{(1+\beta z^{2\zeta})^2}. \tag{2.67}$$

The function $G(z)$ attains maximum value at $z = z_0$, which satisfies $G^{'}(z) = 0$. Solving $G^{'}(z_0)$, we get $z_0 = \sqrt[2\zeta]{\frac{2\zeta-\mu}{\mu}}\beta^{\frac{-1}{2\zeta}}$. Hence,

$$G(z) \le G(z_0) = \frac{\mu}{2\zeta}\left(\frac{2\zeta-\mu}{\mu}\right)^{1-\frac{\mu}{2\zeta}} \beta^{\frac{\mu}{2\zeta}}. \tag{2.68}$$

With $|\omega| \le 1$, we derive the constraint 2.6 as follows:

$$\frac{|R(\beta,\omega)-1|}{|\omega|} = \frac{\beta|\omega|^{2\zeta-1}}{1+\beta|\omega|^{2\zeta}}. \tag{2.69}$$

Let $H(z) = \frac{\beta z^{2\zeta-1}}{1+\beta z^{2\zeta}}$ where $z > 0$. It is easy to see that

$$\lim_{z\to\infty} H(z) = \lim_{z\to\infty} \frac{\beta z^{2\zeta-1}}{1+\beta z^{2\zeta}} = \lim_{z\to\infty} \frac{\beta}{z^{-2\zeta+1}+\beta z} \le \lim_{z\to\infty} \frac{1}{z} = 0. \tag{2.70}$$

$$\lim_{z\to 0} H(z) = \lim_{z\to 0} \frac{\beta z^{2\zeta-1}}{1+\beta z^{2\zeta}} = \lim_{z\to 0} \frac{\beta}{z^{-2\zeta+1}+\beta z} \le \lim_{z\to 0} \frac{\beta}{z^{-2\zeta+1}} = 0. \tag{2.71}$$

By taking the derivative of H with respect to z, we have

$$H'(z) = \frac{((2\zeta-1) - \beta z^{2\zeta})\beta z^{2\zeta-2}}{(1+\beta z^{2\zeta})^2}. \tag{2.72}$$

The function $H(z)$ attains maximum value at $z = z_0$, which satisfies $H^{'}(z) = 0$. Solving $H^{'}(z_0)$, we get $z_0 = \sqrt[2\zeta]{\frac{2\zeta-1}{\beta}}$. Hence,

$$H(z) \le H(z_0) = \frac{(2\zeta-1)^{1-\frac{1}{2\zeta}}}{2\zeta}\beta^{\frac{1}{2\zeta}}. \tag{2.73}$$

□

Remark 2.4. With $R(\beta,\omega)$ and β as in Corollary 3.2 and 3.3 then applying Theorem 2.1, we conclude that the expectation of the error between the regularized solution and the exact solution $\mathbb{E}\|f - 2\widetilde{f}^{\epsilon}\|^2_{L^2(0,1)}$ is of order

$$\max\left\{\epsilon^{\frac{2\lambda\mu}{\zeta}}, \epsilon^{\frac{2\lambda}{\zeta}}, \epsilon^{2-2\lambda}\right\}$$

for any $0 < \lambda < 1$.

3 | NUMERICAL EXPERIMENT

To verify our proposed methods, we carry out numerically the above regularization method. In our computations, we use the Matlab codes for computing the generalized Mittag-Leffler function and the accuracy control in computing is 10^{-8}. In this section, we consider Problem 1.1 with the following exact data

$$\varphi(t) = E_{\alpha,1}(-t^{\alpha}), \quad u_0(x) = \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{6}\right), \quad u_1(x) = E_{\alpha,1}(-1)\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{6}\right), \tag{3.1}$$

TABLE 1 The errors $\epsilon \in \{0.1; 0.01; 0.001\}$ and $\alpha \in \{0.1; 0.5; 0.9\}$

Error(α, σ)	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$
Error$\left(0.1, -\frac{4}{5}\right)$	1.730918352058264	0.176238397738923	0.070161769866725
Error$\left(0.5, -\frac{13}{20}\right)$	2.493076059259953	0.46765811091501	0.014223513375463
Error$\left(0.9, -\frac{12}{25}\right)$	3.761125560996062	1.161580109523819	0.306666254069505

where the fractional order $\alpha = 0, 3$. The corresponding exact solution is given by

$$f(x) = \left(-\frac{x^3}{3} + \frac{x^2}{2} - 2x + \frac{5}{6}\right), \tag{3.2}$$

and

$$u(x,t) = E_{\alpha,1}(-t^\alpha)\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{6}\right). \tag{3.3}$$

It is easy to see that $u(1,t) = 0$ and u satisfies Problem 1.1. The noisy data are generated by adding a random perturbation as follows:

$$\widetilde{u_1}^\epsilon = u_1 + \epsilon\xi. \tag{3.4}$$

We choose $R(\beta, \omega)$ as in Corollary 2.1. Let us define a regularized solution for the system 1.1-1.2 as follows:

$$\widetilde{f}^\epsilon(x) = -\frac{1}{2\pi}\int_{-\omega_{\max}}^{+\omega_{\max}} \frac{\int_0^1 \widetilde{u_1}^\epsilon(x)\cos(\omega x)dx - E_{\alpha,1}(-\omega^2)\int_0^1 u_0(x)\cos(\omega x)dx}{\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\varphi(1-s)ds} e^{i\omega x}d\omega. \tag{3.5}$$

Using,[14] we obtain

$$\int_0^1 E_{\alpha,1}(ys^\alpha)(1-s)^{\beta-1}E_{\alpha,\beta}[z(1-s)^\alpha]ds = \frac{yE_{\alpha,1+\beta}(y) - zE_{\alpha,\beta+1}(z)}{y-z}. \tag{3.6}$$

Let $y = -1$, $\beta = \alpha$ and $z = -\omega^2$ into 3.6, we have

$$\int_0^1 s^{\alpha-1}E_{\alpha,\alpha}(-\omega^2 s^\alpha)\varphi(1-s)ds = \frac{\omega^2 E_{\alpha,\alpha+1}(-\omega^2) - E_{\alpha,\alpha+1}(-1)}{\omega^2 - 1}. \tag{3.7}$$

Hence,

$$\widetilde{f}^\epsilon(x) = -\frac{1}{2\pi}\int_{-\omega_{\max}}^{+\omega_{\max}} \frac{\omega^2 - 1}{\omega^2 E_{\alpha,\alpha+1}(-\omega^2) - E_{\alpha,\alpha+1}(-1)}\left[\int_0^1 \widetilde{u_1}^\epsilon(x)\cos(\omega x)dx - E_{\alpha,1}(-\omega^2)\int_0^1 u_0(x)\cos(\omega x)dx\right] e^{i\omega x}d\omega. \tag{3.8}$$

As in Corollary 2.1, we choose $\omega_{\max} = \epsilon^\sigma$, with $\sigma = -\frac{4}{5}, -\frac{13}{20}, \quad -\frac{12}{25}$, and $\alpha = 0, 1, 0, 5, 0, 9$. The results are shown in Table 1. From these computations, we observe the following property. The regularization methods given in this paper works well for even acceptable error levels. The regularized solution converges to the exact solution with different values of α. However, the numerical accuracy becomes worse as the order of the fractional derivative increases.

ACKNOWLEDGEMENTS

This work is supported by the Institute for Computational Science and Technology (Ho Chi Minh city) under project Grant: 312/QĐ-KHCNTT named "Some ill-posed problems for partial differential equations." The authors also desire to thank the handling editor and the anonymous referees for their helpful comments on this paper.

ORCID

Nguyen Huu Can http://orcid.org/0000-0001-6198-1015

REFERENCES

1. Kirane M, Malik AS, Al-Gwaiz MA. An inverse source problem for a two dimensional time fractional diffusion equation with nonlocal boundary conditions. *Math Methods Appl Sci*. 2013;36(9):1056-1069.
2. Podlubny I. *Fractional Differential Equations Mathematics in Science and Engineering*, Vol. 198. San Diego, CA: Academic Press Inc; 1990.
3. Debnath L. Recent applications of fractional calculus to science and engineering. *Int J Math Math Sci*. 2003;54:3413-3442.
4. Sakamoto K, Yamamoto M. Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J Math Anal Appl*. 2011;382(1):426-447.
5. Zhang ZQ, Wei T. Identifying an unknown source in time fractional diffusion equation by a truncation method. *Appl Math Comput*. 2013;219:5972-5983.
6. Yang F, Fu CL. The quasi-reversibility regularization method for identifying the unknown source for time fractional diffusion equation. *Appl Math Model*. 2015;39(5-6):1500-1512.
7. Wang JG, Zhou YB, Wei T. Two regularization methods to identify a space-dependent source for the time-fractional diffusion equation. *Appl Numer Math*. 2013;68:39-57.
8. Zhang ZQ, Wei T. Identifying an unknown source in time-fractional diffusion equation by a truncation method. *Appl Math Comput*. 2013;219(11):5972-5983.
9. Wei T, Wang J. A modified quasi-boundary value method for an inverse source problem of the time-fractional diffusion equation. *Appl Numer Math*. 2014;78:95-111.
10. Tuan NH, Nane E. Inverse source problem for time-fractional diffusion with discrete random noise. *Statist Probab Lett*. 2017;120:126-134.
11. Cavalier L. Inverse problem in statistics, Heidelberg; April 2007.
12. Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Application of Fractional Differential Equations*: Math Studies Vol. 204. Amsterdam, London and New York: Elsevier (North-Holland) Science Publishers; 2006.
13. Bouchaud JP, Georges A. Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. *Phys Rep*. 1990;195(4-5):127-293.
14. Tuan NH, Kirane M, Hoan LVC, Long LD. Identification and regularization for unknown source for a time-fractional diffusion equation. *Comput Math Appl*. 2017;73(6):931-950.

How to cite this article: Tran Ngoc T, Nguyen Huy T, Pham Thi Minh T, Mach Nguyet M, Nguyen Huu C. Identification of an inverse source problem for time-fractional diffusion equation with random noise. *Math Meth Appl Sci*. 2018;1–15. https://doi.org/10.1002/mma.5334